Definitions Let c be a number in the domain D of a function f. Then $f(c)$ is the

(a) absolute maximum value of f on D, i.e. $f(c) = \max_{x \in D} f(x)$, if $f(c) \geq f(x)$ for all x in D.

(b) absolute minimum value of f on D, i.e. $f(c) = \min_{x \in D} f(x)$, if $f(c) \leq f(x)$ for all x in D.

(c) absolute extremum value of f, if $f(c)$ is either an absolute maximum of minimum value of f on D.

(d) local maximum value of f if $f(c) \geq f(x)$ when $x \in D$ is near c, i.e. there exists $\delta > 0$ such that $f(c) \geq f(x)$ for all $x \in D \cap (c - \delta, c + \delta)$.

(e) local minimum value of f if $f(c) \leq f(x)$ when $x \in D$ is near c, i.e. there exists $\delta > 0$ such that $f(c) \leq f(x)$ for all $x \in D \cap (c - \delta, c + \delta)$.

(f) local extremum value of f if $f(c)$ is either a local maximum or minimum value of f.

(g) A graph is said to be concave up at a point if the tangent line to the graph at that point lies below the graph in the vicinity of the point and concave down at a point if the tangent line lies above the graph in the vicinity of the point.

(h) A point where the concavity changes (from up to down or down to up) is called a point of inflection.

Definitions Let c be a number in the domain D of a function f. Then c is called a critical number of f if either $f'(c) = 0$ or $f'(c)$ does not exist.

Extreme Value Theorem Let f be continuous on $[a, b]$. Then there exist x_1, $x_2 \in [a, b]$ such that

$$\min_{[a,b]} f = f(x_1) \leq f(x) \leq f(x_2) = \max_{[a,b]} f \quad \forall\ x \in [a, b].$$

Fermat's Theorem Let f be continuous on $[a, b]$. If f has a local extremum at $c \in (a, b)$, and if $f'(c)$ exists, then $f'(c) = 0$,
i.e. If f has a local extremum at an interior point $c \in (a, b)$, then either f is not differentiable at c, or f is differentiable at c, and $f'(c) = 0$.

Proof Suppose that

$$\lim_{h \to 0} \frac{f(c + h) - f(c)}{h} = f'(c) > 0,$$

since $\epsilon = \dfrac{f'(c)}{2} > 0$, there exists $\delta > 0$ such that if $0 < |h| < \delta$ then

$$\begin{aligned}
\left|\frac{f(c+h)-f(c)}{h}-f'(c)\right|<\frac{f'(c)}{2} \quad &\Leftrightarrow \quad \frac{f'(c)}{2}>\frac{f(c+h)-f(c)}{h}-f'(c)>-\frac{f'(c)}{2}\\
&\Longrightarrow \quad \frac{f(c+h)-f(c)}{h}>\frac{f'(c)}{2}>0\\
&\Longrightarrow \quad \begin{cases} f(c+h)-f(c)>0 & \text{for all } \delta>h>0\\ f(c+h)-f(c)<0 & \text{for all } 0>h>-\delta \end{cases}\\
&\Leftrightarrow \quad \begin{cases} f(c+h)-f(c)>0 & \text{for all } \delta>h>0\\ f(c-h)-f(c)<0 & \text{for all } \delta>h>0 \end{cases}\\
&\Longrightarrow \quad f(c-h)<f(c)<f(c+h) \quad \text{for all } \delta>h>0\\
&\Longrightarrow \quad f \text{ increases through } c.
\end{aligned}$$

Thus $f(c)$ is not a local extremum of f.

Similarly, if $f'(c)<0$ then f decreases through c and $f(c)$ is not a local extremum of f. Hence, if f has a local extremum at an interior point $c \in (a,b)$ then$f'(c)=0$.

Remark The local extremum of a continuous function f occurs only at a critical number of f.

Remark To find an absolute extremum of a continuous function f on a closed interval $I=[a,b]$, we note that either it is local or it occurs at an endpoint of the interval I.

Rolle's Theorem Let f be continuous on $I=[a,b]$ and let f be differentiable on (a,b). Suppose that $f(a)=f(b)$. Then there is a number $c \in (a,b)$ such that $f'(c)=0$.

Proof
Case 1 Suppose that $f(x)=f(a)=f(b)$ for all $x \in [a,b] \Longrightarrow f'(x)=0$ for all $x \in (a,b)$.

Case 2 Suppose there exits $x \in (a,b)$ such that $f(x) \neq f(a)=f(b) \Longrightarrow$ there exists $c \in (a,b)$ such that $f(c)$ is an extremum value of f. Therefore, $f'(c)=0$ by the Fermat's Theorem.

The Mean Value Theorem Let f be continuous on $I=[a,b]$ and let f be differentiable on (a,b). Then there is a number $c \in (a,b)$ such that

$$f'(c)=\frac{f(b)-f(a)}{b-a} \Longleftrightarrow f(b)-f(a)=f'(c)(b-a).$$

Proof
Consider the function g defined by

$$g(x)=f(x)-f(a)-\frac{f(b)-f(a)}{b-a}(x-a) \quad \text{if } x \in [a,b].$$

Since g is continuous on $I = [a, b]$, differentiable on (a, b) and satisfies that $g(a) = g(b) = 0$, there exists $c \in (a, b)$, by the Rolle's Theorem, such that

$$0 = g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} \Longrightarrow f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Remarks Let f be differentiable on (a, b). Suppose that $f'(x) \neq 0$ for all $x \in (a, b)$. Then

(a) f is $1 - 1$ on (a, b).

Proof For any $x_1 \neq x_2 \in (a, b)$, there exits c lying between x_1 and x_2, by the Mean Value Theorem, such that

$$f(x_1) - f(x_2) = f'(c)(x_1 - x_2) \neq 0 \Longrightarrow f(x_1) \neq f(x_2) \Longrightarrow \ f \text{ is } 1 - 1 \text{ on } (a, b).$$

(b) If $f'(x) > 0$ for all $x \in (a, b)$, then f is increasing on (a, b), i.e. $f(x_1) < f(x_2)$ for any $x_1 < x_2 \in (a, b)$.

Proof For any $x_1 < x_2 \in (a, b)$, there exits $c \in (x_1, x_2)$, by the Mean Value Theorem, such that

$$f(x_1) - f(x_2) = f'(c)(x_1 - x_2) < 0 \Longrightarrow f(x_1) < f(x_2) \Longrightarrow \ f \text{ is increasing on } (a, b).$$

(c) If $f'(x) < 0$ for all $x \in (a, b)$, then f is decreasing on (a, b), i.e. $f(x_1) > f(x_2)$ for any $x_1 < x_2 \in (a, b)$.

Proof For any $x_1 < x_2 \in (a, b)$, there exits $c \in (x_1, x_2)$, by the Mean Value Theorem, such that

$$f(x_1) - f(x_2) = f'(c)(x_1 - x_2) > 0 \Longrightarrow f(x_1) > f(x_2) \Longrightarrow \ f \text{ is decreasing on } (a, b).$$

(d) Let f be twice differentiable on $I = (a, b)$. If $f''(x) > 0$ for all $x \in I = (a, b)$, then f is concave upward on I, i.e. For each $c \in I$, the graph of $y = f(x)$, for x near c, lies above the tangent line to $y = f(x)$ at $(c, f(c))$.

Proof For each $c \in I$, since

$$f''(c) = \lim_{x \to c} \frac{f'(x) - f'(c)}{x - c} > 0,$$

there exists $\delta > 0$ such that if $0 < |x - c| < \delta$ then

$$\begin{aligned} & \frac{f'(x) - f'(c)}{x - c} > 0 \\ \Longrightarrow & \begin{cases} f'(x) - f'(c) > 0 & \forall\, x \in (c, c + \delta) \\ f'(x) - f'(c) < 0 & \forall\, x \in (c - \delta, c). \end{cases} \end{aligned}$$

This imples that if $0 < |x - c| < \delta$, by the Mean Value Theorem, then

$$\begin{aligned} & f(x) - f(c) - f'(c)(x - c) \\ = \ & [f'(z) - f'(c)]\,(x - c) \quad \text{for some } z \text{ lying between } x \text{ and } c \\ > \ & 0 \end{aligned}$$

This proves that the point $(x, f(x))$ in the graph of $y = f(x)$ lies above the point $(x, f(c) + f'(c)(x - c))$ in the tangent line to the graph of $y = f(x)$ at $(c, f(c))$.

(e) Let f be twice differentiable on $I = (a, b)$. If $f''(x) < 0$ for all $x \in I = (a, b)$, then f is concave downward on I, i.e. For each $c \in I$, the graph of $y = f(x)$, for x near c, lies below the tangent line to $y = f(x)$ at $(c, f(c))$.

Cauchy Mean Value Theorem Let $f,\ g$ be continuous on $I = [a, b]$ and let $f,\ g$ be differentiable on (a, b). Then there is a number $c \in (a, b)$ such that

$$f'(c)\,[g(b) - g(a)] = g'(c)[f(b) - f(a)].$$

Proof
Case 1 If $g(b) = g(a)$ then there exists $c \in (a, b)$, by the Rolle's Theorem, such that $g'(c) = 0$. Hence, the equality holds.

Case 2 If $g(b) \neq g(a)$ the function h defined on I by

$$h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}[g(x) - g(a)] \quad \text{if } x \in [a, b].$$

Since h is continuous on $I = [a, b]$, differentiable on (a, b) and satisfies that $h(a) = h(b) = 0$, there exists $c \in (a, b)$, by the Rolle's Theorem, such that

$$0 = h'(c) = f'(c) - \frac{f(b) - f(a)}{g(b) - g(a)}\, g'(c) \Longrightarrow f'(c)\,[g(b) - g(a)] = g'[f(b) - f(a)].$$

An Indeterminate Form $\frac{0}{0}$ and l'Hôpital's Rule Let $f,\ g$ be continuous on $I = [a, b]$ and let $f,\ g$ be differentiable on (a, b). Suppose that

(a) $f(c) = 0 = g(c)$ for some $c \in (a, b)$, i.e. $\lim\limits_{x\to c} f(x) = 0 = \lim\limits_{x\to c} g(x)$,

(b) $g'(x) \neq 0$ for all $x \in (a, c) \cup (c, b)$,

(c) $\lim\limits_{x\to c} \dfrac{f'(x)}{g'(x)} = L \in \mathbb{R}$ exists.

Then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)} = L.$$

Outline of the Proof

$$\begin{aligned}
\lim_{x\to c} \frac{f(x)}{g(x)} &= \lim_{x\to c} \frac{f(x) - f(c)}{g(x) - g(c)} \text{ since } f(c) = g(c) = 0 \\
&= \lim_{x\to c} \frac{f'(t)}{g'(t)} \text{ for some } t \text{ lying between } x \text{ and } c \text{ by the Cauchy Mean Value Theorem} \\
&= \lim_{t\to c} \frac{f'(t)}{g'(t)} \text{ by observng that } t \to c \text{ whenever } x \to c \\
&= L.
\end{aligned}$$

An Indeterminate Form $\pm\frac{\infty}{\infty}$ and l'Hôpital's Rule Let c be a point in the interval (a, b) and let $f,\ g$ be be differentiable on $(a, c) \cup (c, b)$. Suppose that

(a) $\lim\limits_{x\to c} f(x) = \pm\infty$ and $\lim\limits_{x\to c} g(x) = \pm\infty$,

(b) $g'(x) \neq 0$ for all $x \in (a, c) \cup (c, b)$,

(c) $\lim\limits_{x\to c} \dfrac{f'(x)}{g'(x)} = L \in \mathbb{R}$ exists.

Then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)} = L.$$

Outline of the Proof

Case 1 Suppose that $\lim\limits_{x\to c} \dfrac{f'(x)}{g'(x)} = L = 0$.

Given $\epsilon > 0$, since $\lim\limits_{x\to c} \dfrac{f'(x)}{g'(x)} = L = 0$, there exists $\delta_1 > 0$ such that

$$\text{if } 0 < |x - c| < \delta_1 \text{ then } \left|\frac{f'(x)}{g'(x)}\right| < \epsilon.$$

For any points x and x_1 satisfying

$$\text{either } c < x < x_1 < c + \delta_1, \text{ or } c - \delta_1 < x_1 < x < c$$

there exists $x_2 \in (c - \delta_1, c) \cup (c, c + \delta_1)$ lying between x and x_1, by the Cauchy Mean Value Theorem, such that

$$\left|\frac{f(x) - f(x_1)}{g(x) - g(x_1)}\right| = \left|\frac{f'(x_2)}{g'(x_2)}\right| < \epsilon.$$

Also, since

$$\lim_{x\to c} \frac{1 - \dfrac{f(x_1)}{f(x)}}{1 - \dfrac{g(x_1)}{g(x)}} = 1,$$

there exists $\delta_2 > 0$ such that

$$\text{if } 0 < |x - c| < \delta_2 \text{ then } \frac{1 - \dfrac{f(x_1)}{f(x)}}{1 - \dfrac{g(x_1)}{g(x)}} > \frac{1}{2}.$$

Let $\delta = \min\{\delta_1, \delta_2\}$ such that if $0 < |x - c| < \delta$ then

$$\begin{aligned}
\frac{1}{2}\left|\frac{f(x)}{g(x)}\right| &< \left|\frac{f(x)}{g(x)}\left\{\frac{1-\dfrac{f(x_1)}{f(x)}}{1-\dfrac{g(x_1)}{g(x)}}\right\}\right| \\
&= \left|\frac{f(x)-f(x_1)}{g(x)-g(x_1)}\right| \\
&= \left|\frac{f'(x_2)}{g'(x_2)}\right| \\
&< \epsilon
\end{aligned}$$

which implies that

$$\left|\frac{f(x)}{g(x)}\right| < 2\epsilon.$$

Hence,

$$\lim_{x\to c}\frac{f(x)}{g(x)} = 0 = L = \lim_{x\to c}\frac{f'(x)}{g'(x)}.$$

Case 2 Suppose that $\lim_{x\to c}\dfrac{f'(x)}{g'(x)} = L \neq 0$.
Consider the function h defined by $h(x) = f(x) - L\,g(x)$ for all $x \in (a,c)\cup(c,b)$ and note that

$$\lim_{x\to c}\frac{h'(x)}{g'(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)} - L = 0.$$

Applying **Case** 1, we get

$$\lim_{x\to c}\frac{f(x)}{g(x)} - L = \lim_{x\to c}\frac{h(x)}{g(x)} = 0 \Longrightarrow \lim_{x\to c}\frac{f(x)}{g(x)} = L.$$